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Computing All Power Integral Bases of Cubic Fields

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Abstract. Applying Baker's effective method and the reduction procedure of Baker and Davenport, we present several lists of solutions of index form equations in (totally real and complex) cubic algebraic number fields. These solutions yield all power integral bases of these fields.

1. Introduction. Let K be a cubic algebraic number field and denote by Z_K the ring of integers of K. A power integral basis of K is an integral basis of the form $\{1, \alpha, \alpha^2\}$ with some $\alpha \in Z_K$. If there exists such an α, then we say that K is monogenic, since $Z_K = Z[\alpha]$. Obviously, if α has this property, then it holds also for $\alpha + k$ with any $k \in Z$.

From a practical point of view, it is important to know whether there exists a power integral basis of K, and if so, what are the numerical values of α.

If the Galois group of K is cyclic, then the discriminant of K is a full square. The problem of monogeneity in cyclic cubic fields was considered by M. N. Gras [12], [13], Archinard [1] and Dummit and Kisilevsky [4]. M. N. Gras and Archinard gave necessary and sufficient conditions for monogeneity and tested for several numerical examples whether or not the field is monogenic. Moreover, Dummit and Kisilevsky proved that there exist infinitely many cyclic cubic fields with power integral bases.

For arbitrary algebraic number fields L, it was proved by Györy [14] that up to obvious translations by elements of Z, there are only finitely many $\alpha \in Z_L$ with $Z_L = Z[\alpha]$, and he gave effective (but rather large) bounds for the sizes of these α.

In this paper we present a method which allows us to determine all possible values of α (up to translation with rational integers) such that $\{1, \alpha, \alpha^2\}$ is a power integral basis of a given cubic number field K. Let $\{w_1 = 1, w_2, w_3\}$ be an integral basis of K. Denote by $\beta^{(i)}$ $(i = 1, 2, 3)$ the conjugates of any $\beta \in K$. The discriminant of the linear form $w_2X + w_3Y$ can be written as

$$\begin{aligned} D_{K/Q}(w_2X + w_3Y) &= \prod_{1 \le i < j \le 3} ((w_2^{(i)} - w_2^{(j)})X + (w_3^{(i)} - w_3^{(j)})Y)^2 \\ &= (I(X, Y))^2 D, \end{aligned}$$

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where D is the discriminant of the field K and $I(X,Y)$ is a homogeneous cubic polynomial with rational integral coefficients, which is called the *index form* with respect to the basis $\{w_1 = 1, w_2, w_3\}$ of K. The equation above yields that $\{1, \alpha, \alpha^2\}$ ($\alpha = xw_2 + yw_3$) is an integral basis of the field K if and only if $D_{K/Q}(\alpha) = D$, that is, if (x,y) is a solution of the *index form equation*

$$I(x,y) = \pm 1 \qquad (x, y \in Z). \tag{1}$$

In the special case of cubic number fields, the index form equation (1) is just a cubic Thue equation. Using Baker's method, effective bounds for the solutions of index form equations (corresponding to arbitrary algebraic number fields) were obtained by Győry [14]. His result was later improved and generalized by Győry and Papp [17], Trelina [25] and Győry [15], [16]. These bounds imply, in principle, that the solutions can be determined, but the bounds are too high for practical applications. On the other hand, an idea of Baker and Davenport [2] makes it possible to reduce the large upper bounds. Applying this method, Ellison [7], Ellison et al. [8], Steiner [24] and Pethö and Schulenberg [23] solved completely certain Thue equations of degree three and four. Moreover, Pethö [21] worked out a fast method to find "small" solutions of Thue equations.

2. Brief Sketch of the Algorithm. Besides new ideas, our method also involves some standard arguments (used by the authors quoted above). We therefore do not go into details describing those algorithms; we shall only recall the main steps of our algorithm, point out the differences between the real and complex cases and stress the new features in comparison with equations solved by other authors.

Let the index form equation (1) (corresponding to the integral basis $\{1, w_2, w_3\}$) of the cubic field K be

$$I(x,y) = I_3x^3 + I_2x^2y + I_1xy^2 + I_0y^3 = \pm 1 \qquad (x, y \in Z). \tag{2}$$

Denote by τ a root of $I(x,1) = 0$. We remark that τ can be chosen to generate (the same field) K over Q. Let $x, y \in Z$ be an arbitrary but fixed solution of (2) and put $\beta = I_3(x - \tau y)$. β is an integer of K, and (2) can be written as

$$\beta^{(1)}\beta^{(2)}\beta^{(3)} = \pm I_3^2.$$

In the real case, denote by η_1, η_2 (resp. by η_1 in the complex case) the fundamental units of K with norm $+1$. Then we set

$$\beta = \gamma\eta_1^{b_1}\eta_2^{b_2} \quad (\text{resp.}\, \beta = \gamma\eta_1^{b_1}), \tag{3}$$

where $b_1, b_2 \in Z$ (resp. $b_1 \in Z$) and γ is an integral element in K with norm $\pm I_3^2$. If $|I_3| \neq 1$, then in the sequel we consider separately each element of a full set of nonassociate elements γ of norm I_3^2. Such a set can be determined, e.g., by the method of Fincke and Pohst [10]. We remark that this is the first time that Thue equations with $|I_3| \neq 1$ are solved completely.

Denote by k the index with $|\beta^{(k)}| = \min|\beta^{(i)}|$ (the minimum is taken for $i = 1,2,3$) and let $\{i,j\} = \{1,2,3\}\backslash\{k\}$. In the real case, we have to consider $k = 1,2,3$ separately, but in the complex case, the only interesting case is when $\tau^{(k)}$ is the real conjugate of τ (cf. [21]).

Applying Siegel's identity and some standard estimates, in the real case we obtain that, if $B = \max(|b_1|, |b_2|)$ is large enough,

$$|b_1 \log|\delta_1| + b_2 \log|\delta_2| - \log|\delta_3|| < \exp(c_1 - c_2 B). \tag{4}$$

In the complex case, if $|b_1|$ is not too small, then the corresponding inequality is

$$|b_1 \log(\delta_1) + b_2 \log(-1) - \log(\delta_2)| < \exp(c_1 - c_2 B), \tag{5}$$

where log denotes the principal value, $b_2 \in Z$ with $|b_2| \leq 2|b_1|$ and $B = \max(|b_1|, |b_2|)$. In both cases, $\delta_1, \delta_2, \delta_3$ are algebraic numbers in the Galois closure of K depending on γ, the fundamental unit(s), τ, and the indices k, i, j. Furthermore, $c_1, c_2, \ldots$ denote explicitly given positive constants.

Applying Baker's method and the better estimates of Waldschmidt [26] (for the sake of getting sharp constants), in both cases the respective linear forms can be estimated from below by a constant of the form

$$\exp(-c_3(\log B + c_4)).$$

Comparing the lower and upper estimates obtained for the linear forms (4), (5), we get an upper bound B_u for B, which is about 10^{27}, 10^{28} in the cases we computed.

Dividing (4) and (5) by the coefficient of b_2 (in both cases), we obtain

$$|b_1 \theta + b_2 - \xi| < c_5 c_6^{-B}, \tag{6}$$

where θ, ξ are the quotients of two logarithms. This is the inequality to which the Baker-Davenport reduction method can be applied. For our case, a suitable version of the lemma is formulated in [11]. The essence of it is that, if we can find a good approximation of θ, which does not approximate well ξ, then (6) has no solutions b_1, b_2 with

$$\frac{c_7 + \log B_u}{c_8} < B < B_u,$$

where c_7, c_8 have moderate values. Thus, we can reduce B_u almost to its logarithm, and we get a much better upper bound for B. Repeating the reduction step three or four times (until the new bound is still less than the original one), we get quite a low bound for B, which, in our computation, was usually ≤ 12. We remark that in the first reduction step it is necessary to use multiple precision arithmetic and to calculate the numerical values for θ and ξ with an accuracy of about 100 digits.

Using the reduced bound for B, from (3) it is easy to derive a bound y_0 for $|y|$, and the procedure is completed by applying a fast algorithm of Pethö [21] which helps to find all solutions of (1) with $|y| < y_0$.

3. Computational Results. Using the above method, we solved index form equations (1) corresponding to cubic number fields with discriminants $-300 \leq D \leq 3137$.

From a computational point of view, the totally real case is more interesting than the complex case. This is why we included more examples with positive discriminants. Further, let us remark that in the complex case, much more is known about the number of solutions of (1) than in the real case. To formulate the corresponding theorem, recall that two cubic forms f_1, $f_2 \in Z[X, Y]$ are called *equivalent* if there exist integers a_1, a_2, a_3, a_4 with $a_1 a_4 - a_2 a_3 = \pm 1$ such that

$f_2(X,Y) = f_1(a_1X + a_2Y, a_3X + a_4Y)$. A form $f(X,Y) = aX^3 + bX^2Y + cXY^2 + dY^3$ is *reversible* if $a = d = 1$ (cf. [6]).

Denote by N the number of solutions of

$$f(x,y) = 1 \quad \text{in } x, y \in Z \tag{7}$$

with the above f. Our equation (1) is precisely of the type (7), hence it is interesting to compare our results with those previously known about N. In the complex case, Delone [5] and Nagell [20] proved that $N \le 5$, and that

$$N = \begin{array}{rcl} \text{if } f \text{ is not equivalent} & \left\{\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right. & \\ \text{to a reversible form} & \left.\begin{array}{c} 2 \\ 3 \end{array}\right\} & \begin{array}{l}\text{if } f \text{ is equivalent to} \\ \text{a reversible form}\end{array} \\ & 4 & \text{if } D = -44 \text{ and } -31 \\ & 5 & \text{if } D = -23, \end{array}$$

and there exist infinitely many inequivalent forms with $N = 3$.

The corresponding assertion in the real case has not yet been verified, but our results make probable a conjecture of Pethö [22], namely that $N \le 9$ and that

$$N = \begin{array}{rcl} \text{if } f \text{ is not equivalent} & \left\{\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}\right. & \\ \text{to a reversible form} & \left.\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \end{array}\right\} & \begin{array}{l}\text{if } f \text{ is equivalent to} \\ \text{a reversible form}\end{array} \\ & 6 & \text{if } D = 81, 229, 257, 361 \\ & 7 & \text{none} \\ & 8 & \text{none} \\ & 9 & \text{if } D = 49. \end{array}$$

We remark that the numerical data of all power integral bases may have several applications, apart from using them in calculations in these fields. For example, Kovács [18] proved that the so-called canonical number systems of number fields (cf. [18]) are closely connected with power integral bases. Our data were used by Kovács and Pethö [19] to compute all canonical number systems in certain cubic number fields. Further, our table makes it possible to find several exceptional units in modules $\{1, \alpha\}$ $(\alpha \in Z_K)$ of cubic number fields.

The numerical results are listed in the table below. For every example, the table contains the following data:

$$D \quad (a_2, a_1, a_0) \quad (I_3, I_2, I_1, I_0) \quad (x_1, y_1), (x_2, y_2), \ldots,$$

where D is the discriminant of the field K, $f(x) = x^3 + a_2x^2 + a_1x + a_0$ is the defining polynomial of the generator element γ of K over Q. The index form is

$$I(X,Y) = I_3X^3 + I_2X^2Y + I_1XY^2 + I_0Y^3,$$

and the solutions of (1) are (x_1, y_1), $(x_2, y_2), \ldots$. Obviously, if (x, y) is a solution of (1), then so is $(-x, -y)$, but we include only one of them in the table. If the given integral basis (corresponding to the index form) is different from $\{1, \gamma, \gamma^2\}$, and it has the form $\{1, w_2, w_3\}$ with

$$w_2 = (p_0 + p_1\gamma + p_2\gamma^2)/p, \qquad w_3 = (q_0 + q_1\gamma + q_2\gamma^2)/q,$$

then we also add the data

$$w2 = (p_0, p_1, p_2)/p, \qquad w3 = (q_0, q_1, q_2)/q.$$

The input parameters (discriminant, defining polynomial of the generator element, integral basis, fundamental units) were taken in the real case from Ennola and Turunen [9], and in the complex case from Buchmann [3].

The computer program was developed in Fortran and was executed on the Siemens 7570P computer of the University of Düsseldorf. The execution time was about 60 seconds for every example.

TABLE

Real cubic fields

D	$f(x)$	$I(X,Y)$	solutions
49	$(-1,-2,1)$	$(1,2,-1,-1)$	$(-2,1),(-9,4),(-1,1),(-1,-1),(-1,2),$ $(-5,9),(0,1),(4,5),(1,0)$
81	$(0,-3,1)$	$(1,0,-3,1)$	$(-2,1),(1,1),(3,2),(0,1),(1,3),(1,0)$
148	$(-1,-3,1)$	$(1,2,-2,-2)$	$(1,1),(1,-1),(-5,2),(-31,45),(1,0)$
169	$(-1,-4,-1)$	$(1,2,-3,-5)$	$(-2,1),(-1,1),(1,0)$
229	$(0,-4,1)$	$(1,0,-4,1)$	$(-2,1),(-2,-1),(508,273),(0,1),(1,4)$, $(1,0)$
257	$(-1,-4,3)$	$(1,2,-3,-1)$	$(1,1),(6,5),(-3,1),(0,1),(-2,7),(1,0)$
316	$(-1,-4,2)$	$(1,2,-3,-2)$	$(-1,2),(1,0)$
321	$(-1,-4,1)$	$(1,2,-3,-3)$	$(1,-1),(1,0)$
361	$(-1,-6,7)$	$(1,2,-5,1)$	$(-7,2),(1,1),(9,7),(0,1),(2,9),(1,0)$
404	$(-1,-5,-1)$	$(1,2,-4,-6)$	$(1,-1),(1,0)$
469	$(-1,-5,4)$	$(1,2,-4,-1)$	$(0,1),(1,0)$
473	$(0,-5,1)$	$(1,0,-5,1)$	$(-2,-1),(-7,3),(0,1),(1,5),(1,0)$
564	$(-1,-5,3)$	$(1,2,-4,-2)$	$(3,2),(-3,1),(-3,7),(1,0)$
568	$(-1,-6,-2)$	$(1,2,-5,-8)$	$(17,8),(1,0)$
621	$(0,-6,3)$	$(1,0,-6,3)$	$(-8,3),(2,1),(1,2),(1,0)$
697	$(0,-7,5)$	$(1,0,-7,5)$	$(-3,1),(2,1),(13,6),(1,1),(1,0)$
733	$(-1,-7,8)$	$(1,2,-6,1)$	$(3,2),(0,1),(1,0)$
756	$(0,-6,2)$	$(1,0,-6,2)$	$(1,3),(1,0)$
761	$(-1,-6,-1)$	$(1,2,-5,-7)$	$(2,1),(-3,1),(-1,1),(1,0)$
785	$(-1,-6,5)$	$(1,2,-5,-1)$	$(0,1),(1,0)$
788	$(-1,-7,-3)$	$(1,2,-6,-10)$	$(-3,1),(-3,2),(1,0)$
837	$(0,-6,1)$	$(1,0,-6,1)$	$(0,1),(1,6),(1,0)$
892	$(-1,-8,10)$	$(1,2,-7,2)$	$(1,0)$
940	$(0,-7,4)$	$(1,0,-7,4)$	$(1,0)$
961	$(-1,-10,8)$	$(2,5,-1,-2)$	no solutions
	$w2 = (0,1,0)/1$	$w3 = (0,1,1)/2$	
985	$(-1,-6,1)$	$(1,2,-5,-5)$	$(1,-1),(2,1),(-3,1),(1,0)$
993	$(-1,-6,3)$	$(1,2,-5,-3)$	$(-1,2),(1,0)$
1016	$(-1,-6,2)$	$(1,2,-5,-4)$	$(1,0)$
1076	$(0,-8,6)$	$(1,0,-8,6)$	$(7,3),(1,1),(1,0)$
1101	$(-1,-9,12)$	$(1,2,-8,3)$	$(1,0)$
1129	$(0,-7,3)$	$(1,0,-7,3)$	$(1,0)$
1229	$(-1,-7,6)$	$(1,2,-6,-1)$	$(7,4),(0,1),(1,0)$
1257	$(-1,-8,9)$	$(1,2,-7,1)$	$(0,1),(1,0)$
1300	$(0,-10,10)$	$(1,0,-10,10)$	$(1,1),(1,0)$
1345	$(0,-7,1)$	$(1,0,-7,1)$	$(-19,7),(18,7),(0,1),(1,7),(1,0)$
1369	$(-1,-12,-11)$	$(1,2,-11,-23)$	$(3,-1),(-2,1),(1,0)$

TABLE (*continued*)

D	$f(x)$	$I(X,Y)$	solutions
1373	$(0,-8,5)$	$(1,0,-8,5)$	$(2,3),(1,0)$
1384	$(-1,-10,14)$	$(1,2,-9,4)$	$(25,14),(1,2),(1,0)$
1396	$(-1,-7,5)$	$(1,2,-6,-2)$	$(1,0)$
1425	$(-1,-8,-3)$	$(1,2,-7,-11)$	$(3,-1),(1,0)$
1436	$(0,-11,12)$	$(1,0,-11,12)$	$(5,2),(1,0)$
1489	$(-1,-10,-7)$	$(1,2,-9,-17)$	$(2,-1),(3,1),(3,-1),(-22,7),(1,0)$
1492	$(-1,-9,-5)$	$(1,2,-8,-14)$	$(3,-1),(1,0)$
1509	$(-1,-7,4)$	$(1,2,-6,-3)$	$(2,1),(1,0)$
1524	$(-1,-7,1)$	$(1,2,-6,-6)$	$(-1,1),(1,0)$
1556	$(-1,-9,11)$	$(1,2,-8,2)$	$(1,0)$
1573	$(-1,-7,2)$	$(1,2,-6,-5)$	$(2,1),(-13,18),(1,0)$
1593	$(0,-9,7)$	$(1,0,-9,7)$	$(-10,3),(5,2),(1,1),(1,0)$
1620	$(0,-12,14)$	$(1,0,-12,14)$	$(1,0)$
1708	$(-1,-8,-2)$	$(1,2,-7,-10)$	$(1,0)$
1765	$(-1,-11,16)$	$(1,2,-10,5)$	$(2,1),(1,0)$
1772	$(-1,-12,8)$	$(2,5,-2,-3)$	$(-8,3),(-2,3)$
	$w2=(0,1,0)/1$	$w3=(0,1,1)/2$	
1825	$(-1,-8,7)$	$(1,2,-7,-1)$	$(2,1),(0,1),(1,0)$
1849	$(-1,-14,-8)$	$(2,5,-3,-8)$	no solutions
	$w2=(0,1,0)/1$	$w3=(0,1,1)/2$	
1901	$(-1,-9,-4)$	$(1,2,-8,-13)$	$(-42,13),(-3,2),(1,0)$
1929	$(-1,-10,13)$	$(1,2,-9,3)$	$(2,1),(1,0)$
1937	$(-1,-8,-1)$	$(1,2,-7,-9)$	$(-1,1),(1,0)$
1940	$(0,-8,2)$	$(1,0,-8,2)$	$(-3,1),(1,4),(1,0)$
1944	$(0,-9,6)$	$(1,0,-9,6)$	$(1,0)$
1957	$(-1,-9,10)$	$(1,2,-8,1)$	$(-4,1),(2,1),(0,1),(1,0)$
2021	$(0,-8,1)$	$(1,0,-8,1)$	$(-26,9),(0,1),(1,8),(1,0)$
2024	$(-1,-10,-6)$	$(1,2,-9,-16)$	$(1,0)$
2057	$(0,-11,11)$	$(1,0,-11,11)$	$(1,1),(1,0)$
2089	$(0,-13,4)$	$(2,3,-5,-2)$	no solutions
	$w2=(0,1,0)/1$	$w3=(0,1,1)/2$	
2101	$(-1,-11,-8)$	$(1,2,-10,-19)$	$(-2,1),(1,0)$
2177	$(-1,-8,5)$	$(1,2,-7,-3)$	$(2,1),(1,0)$
2213	$(-1,-13,-12)$	$(1,2,-12,-25)$	$(-2,1),(1,0)$
2233	$(-1,-8,1)$	$(1,2,-7,-7)$	$(-1,1),(1,0)$
2241	$(0,-9,5)$	$(1,0,-9,5)$	$(8,3),(1,0)$
2296	$(-1,-14,-14)$	$(1,2,-13,-28)$	$(-9,4),(1,0)$
2300	$(-1,-8,2)$	$(1,2,-7,-6)$	$(-7,2),(1,0)$
2349	$(0,-12,13)$	$(1,0,-12,13)$	$(8,3),(1,0)$
2505	$(-1,-10,-5)$	$(1,2,-9,-15)$	$(1,0)$
2557	$(-1,-9,-2)$	$(1,2,-8,-11)$	$(1,0)$
2589	$(-1,-14,12)$	$(2,5,-3,-3)$	$(1,1)$
	$w2=(0,1,0)/1$	$w3=(0,1,1)/2$	
2597	$(-1,-9,8)$	$(1,2,-8,-1)$	$(2,1),(-4,1),(0,1),(1,0)$
2636	$(0,-14,4)$	$(2,0,-7,1)$	$(-2,1),(0,1)$
	$w2=(0,1,0)/1$	$w3=(0,0,1)/2$	
2673	$(0,-9,3)$	$(1,0,-9,3)$	$(1,3),(1,0)$
2677	$(0,-10,7)$	$(1,0,-10,7)$	$(1,0)$
2700	$(0,-15,20)$	$(1,0,-15,20)$	$(1,0)$
2708	$(-1,-11,-7)$	$(1,2,-10,-18)$	$(1,0)$
2713	$(0,-13,15)$	$(1,0,-13,15)$	$(-49,12),(4,3),(1,0)$
2777	$(-1,-14,23)$	$(1,2,-13,9)$	$(-5,1),(2,1),(17,8),(1,1),(1,0)$
2804	$(-1,-9,-1)$	$(1,2,-8,-10)$	$(-1,1),(1,0)$
2808	$(0,-9,2)$	$(1,0,-9,2)$	$(1,0)$
2836	$(-1,-9,7)$	$(1,2,-8,-2)$	$(1,0)$
2857	$(-1,-10,11)$	$(1,2,-9,1)$	$(-21,5),(2,1),(1,0),(0,1)$
2917	$(-1,-13,20)$	$(1,2,-12,7)$	$(2,1),(1,0)$

TABLE (*continued*)

D	$f(x)$	$I(X,Y)$	solutions
2981	$(-1,-11,14)$	$(1,2,-10,3)$	$(2,1),(1,0)$
2993	$(-1,-12,17)$	$(1,2,-11,5)$	$(2,1),(1,2),(1,0)$
3021	$(-1,-9,6)$	$(1,2,-8,-3)$	$(1,0)$
3028	$(0,-10,6)$	$(1,0,-10,6)$	$(1,0)$
3137	$(0,-11,9)$	$(1,0,-11,9)$	$(-11,3),(1,1)$

Complex cubic fields

D	$f(x)$	$I(X,Y)$	solutions
-23	$(0,-1,1)$	$(1,0,-1,1)$	$(-1,1),(0,1),(1,0),(1,1),(4,-3)$
-31	$(0,1,1)$	$(1,0,1,1)$	$(-2,3),(0,1),(1,-1),(1,0)$
-44	$(1,-1,1)$	$(1,-2,0,2)$	$(-47,56),(1,-1),(1,0),(1,1)$
-59	$(0,2,1)$	$(1,0,2,1)$	$(0,1),(1,-2),(1,0)$
-76	$(0,-2,2)$	$(1,0,-2,2)$	$(-23,13),(1,0),(1,1)$
-83	$(1,1,2)$	$(1,-2,2,1)$	$(0,1),(1,0)$
-87	$(1,-2,-3)$	$(1,-2,-1,-1)$	$(0,-1),(1,0)$
-104	$(0,-1,2)$	$(1,0,-1,2)$	$(-3,2),(1,0)$
-107	$(1,3,2)$	$(1,-2,4,-1)$	$(0,-1),(1,0),(2,7)$
-108	$(0,0,2)$	$(1,0,0,2)$	$(-1,1),(1,0)$
-116	$(1,0,2)$	$(1,-2,1,2)$	$(1,0)$
-135	$(0,-3,3)$	$(1,0,-3,3)$	$(-2,1),(1,0),(1,1)$
-139	$(1,1,-2)$	$(1,-2,2,-3)$	$(1,0),(2,1)$
-140	$(0,2,2)$	$(1,0,2,2)$	$(1,-1),(1,0)$
-152	$(1,-2,2)$	$(1,-2,-1,4)$	$(1,0)$
-172	$(1,-1,-3)$	$(1,-2,0,-2)$	$(1,0)$
-175	$(1,2,3)$	$(1,-2,3,1)$	$(0,1),(1,0)$
-199	$(1,-4,3)$	$(1,-2,-3,7)$	$(1,0),(2,1)$
-200	$(1,2,-2)$	$(1,-2,3,-4)$	$(1,0)$
-204	$(1,1,3)$	$(1,-2,2,2)$	$(1,0)$
-211	$(0,-2,3)$	$(1,0,-2,3)$	$(1,0),(2,-1)$
-212	$(1,4,2)$	$(1,-2,5,-2)$	$(1,0),(1,2)$
-216	$(0,3,2)$	$(1,0,3,2)$	$(1,0)$
-231	$(1,0,-3)$	$(1,-2,1,-3)$	$(-2,-1),(1,0)$
-239	$(0,-1,3)$	$(1,0,-1,3)$	$(-5,3),(1,0)$
-243	$(0,0,3)$	$(1,0,0,3)$	$(1,0)$
-244	$(1,-4,-6)$	$(1,-2,-3,-2)$	$(1,0)$
-247	$(0,1,3)$	$(1,0,1,3)$	$(-1,1),(1,0)$
-255	$(1,0,3)$	$(1,-2,1,3)$	$(1,-1),(1,0)$
-268	$(1,-3,-5)$	$(1,-2,-2,-2)$	$(1,0),(3,1)$
-283	$(0,4,1)$	$(1,0,4,1)$	$(0,1),(1,-4),(1,0)$
-300	$(1,-3,3)$	$(1,-2,-2,6)$	$(1,0)$

After the complete list of cubic fields with discriminants $-300 \le D \le 3137$ we give some more examples which may be of interest.

D	$f(x)$	$I(X,Y)$	solutions
22356	$(0,-36,60)$	$(2,0,-18,15)$	$(1,1)$
	$w2=(0,1,0)/1$	$w3=(0,0,1)/2$	
22356	$(0,-18,6)$	$(1,0,-18,6)$	$(1,3),(1,0)$
22356	$(0,-36,78)$	$(1,0,-36,78)$	$(485,117),(1,0)$
677329	$(1,-274,61)$	$(11,-14,-19,11)$	no solutions
	$w2=(0,1,0)/1$	$w3=(-1,-4,1)/11$	

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